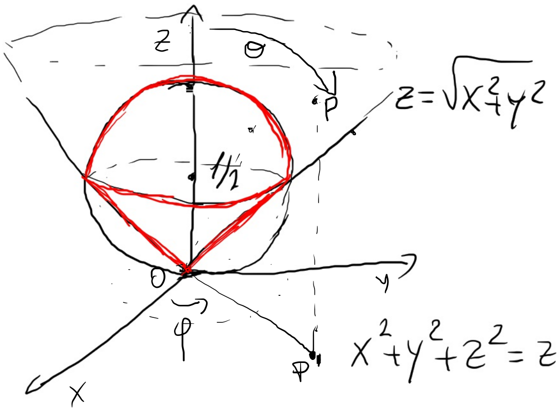
Dú 7

$P(\varrho, \varphi, \vartheta)$

$$x = \varrho \sin\vartheta \cos\varphi$$
$$y = \varrho \sin\vartheta \sin\varphi \qquad \varrho \geq 0,\ \varphi \in \langle 0, 2\pi \rangle,\ \vartheta \in \langle 0, \pi \rangle$$
$$z = \varrho \cos\vartheta.$$

Nalezněte objem tělesa M ohraničeného plochami $z = x^2 + y^2 + z^2$, a $z = \sqrt{x^2 + y^2}$.

$$x^2 + y^2 + z^2 - z = 0$$
$$x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4}$$



$x^2 + y^2 + z^2 = z \rightsquigarrow$ sf. $\quad \varrho^2 = \varrho \cos\vartheta$

$\varrho = \cos\vartheta$

$$\text{Objem}(M) = \iiint_M dV = \text{sfer.}$$

$$= \int_0^{\pi/4} \int_0^{2\pi} \int_0^{\cos\vartheta} \varrho^2 \sin\vartheta \, d\varrho \, d\varphi \, d\vartheta =$$

$$= \int_0^{\pi/4} \int_0^{2\pi} \frac{\cos^3\vartheta}{3} \sin\vartheta \, d\varphi \, d\vartheta =$$

$$= 2\pi \left[-\frac{\cos^4\vartheta}{12} \right]_0^{\pi/4} =$$

$$= \frac{\pi}{6} \left[-\left(\frac{\sqrt{2}}{2}\right)^4 + 1 \right] = \frac{\pi}{8}.$$

5 Vypočtěte

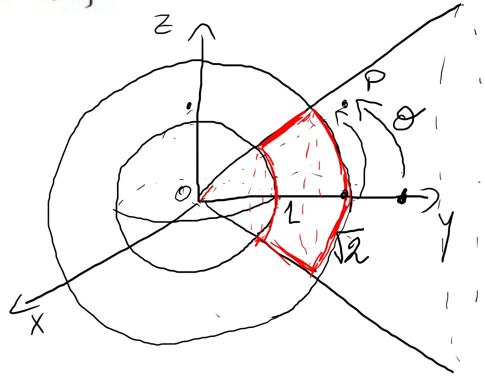
$$\iiint_E \frac{x^2}{x^2+z^2}\,dV,$$

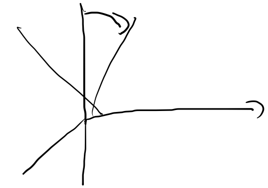
kde $E = \{(x,y,z) : 1 \le x^2+y^2+z^2 \le 2,\ y > 0,\ x^2-y^2+z^2 < 0\}$.

$$x^2 - y^2 + z^2 = 0$$

$$y^2 = x^2 + z^2$$

$$\begin{cases} y = \pm\sqrt{x^2+z^2} \\ y > 0 \qquad y = \sqrt{x^2+z^2} \end{cases}$$





$$\Phi \begin{cases} y = \rho\cos\theta \\ x = \rho\sin\theta\cos\varphi \\ z = \rho\sin\theta\sin\varphi \end{cases}$$

$$|J_\Phi| = \rho^2 \sin\theta$$

$$\int_0^{2\pi}\int_0^{\pi/4}\int_1^{\sqrt{2}} \frac{\cancel{\rho^2}\sin^{\cancel{2}}\theta\cos^2\varphi}{\cancel{\rho^2}\sin^{\cancel{2}}\theta}\rho^2\sin\theta\, d\rho\, d\theta\, d\varphi =$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\int_1^{\sqrt{2}} \rho^2\cos^2\varphi\sin\theta\, d\rho\, d\theta\, d\varphi =$$

$$= \int_0^{2\pi} \frac{1+\cancel{\cos 2\varphi}_{0}}{2}\, d\varphi \cdot \int_0^{\pi/4} \sin\theta\, d\theta \cdot \int_1^{\sqrt{2}} \rho^2\, d\rho =$$

$$= \frac{1}{2}\cdot 2\pi \cdot \left[-\cos\theta\right]_0^{\pi/4} \cdot \left[\frac{\rho^3}{3}\right]_1^{\sqrt{2}} = \pi\cdot\left(-\frac{\sqrt{2}}{2}+1\right)\cdot\frac{2\sqrt{2}-1}{3}.$$

Délka křivky, křivkový integrál.

Délka křivky Γ s parametrizací φ se vypočítá jako $\ell(\Gamma) = \int_a^b ||\varphi'(t)||\ dt \ \left(= \int_\Gamma 1\ ds\right)$.

$\varphi(t) : [a,b] \to \mathbb{R}^n$

$[a,b] \subseteq \mathbb{R}$

Určete délku cykloidy Γ s parametrizací

$$\varphi(t) = (\overbrace{t - \sin t}^{x}, \overbrace{1 - \cos t}^{y}) \qquad \underset{a}{0} \le t \le \underset{b}{2\pi}.$$

$$\varphi'(t) = \Big(1 - \cos t,\ \sin t\Big) \quad \text{a} \quad ||\varphi'(t)|| = \sqrt{(1-\cos t)^2 + \sin^2 t} = \sqrt{2 - 2\cos t}.$$

$1 + \cos^2 t - 2\cos t + \sin^2 t$

$$\ell(\Gamma) = \int_\Gamma 1\ ds = \sqrt{2}\int_0^{2\pi} \sqrt{1-\cos t}\ dt = \left[\begin{smallmatrix} 2u = t \\ 2du = dt \end{smallmatrix}\right] = 2\sqrt{2}\int_0^{\pi} \sqrt{1-\cos(2u)}\ du =$$

$$= \left\{\cos(2u) = \cos^2 u - \sin^2 u\right\} = 4\int_0^{\pi} \sin u\ du = 8\ .$$

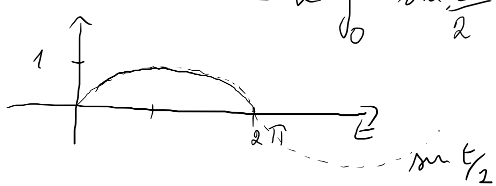
$$\cos^2 t = \frac{1 + \cos 2t}{2}$$

$$\sin^2 t = \frac{1 - \cos 2t}{2}$$

$$\sqrt{2}\int_0^{2\pi} \sqrt{2\sin^2 t/2}\ dt = 2\int_0^{2\pi} \left|\sin \frac{t}{2}\right| dt$$

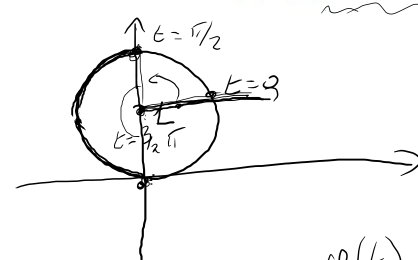
$$= 2\int_0^{2\pi} \sin \frac{t}{2}\ dt = 2\left(-\cos\frac{t}{2}\right)_0^{2\pi} \cdot 2 = 4 + 4 = 8\ .$$

$$\sqrt{\sin^2 u} = |\sin u|$$



Integrál z funkce f podél křivky $\mathcal{C}$ spočítáme podle vztahu $\int\limits_{\mathcal{C}} f \; ds = \int\limits_a^b f(\varphi(t)) \cdot ||\varphi'(t)|| \; dt$, kde φ je vhodná parametrizace křivky $\mathcal{C}$.

2 Spočítejte $\int\limits_{\mathcal{C}} (x+y)\, ds$, kde $\mathcal{C}$ je levá polovina kružnice o poloměru 1 se středem v $(0,1)$ jdoucí v záporném smyslu z bodu $(0,0)$ do bodu $(0,2)$.



$$\varphi(t) = (\overbrace{\cos t}^{x}, \overbrace{1+\sin t}^{y}) \qquad t \in \langle \pi/2, \tfrac{3}{2}\pi \rangle$$

$$\varphi'(t) = (-\sin t, \; \cos t)$$

$$\|\varphi'(t)\| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

$$\int_C (x+y)\, ds = \int_{\pi/2}^{3/2\pi} (\cos t + 1 + \sin t)\, 1 \, dt =$$

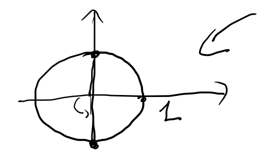
$$= \left[\sin t + t - \cos t\right]_{\pi/2}^{3/2\pi} = \ldots\ldots$$

$\varphi(t)$

$t \in [a,b]$

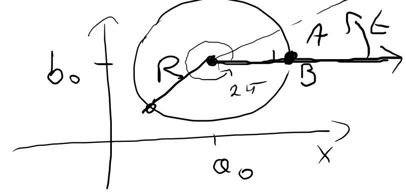
$A = \varphi(a)$

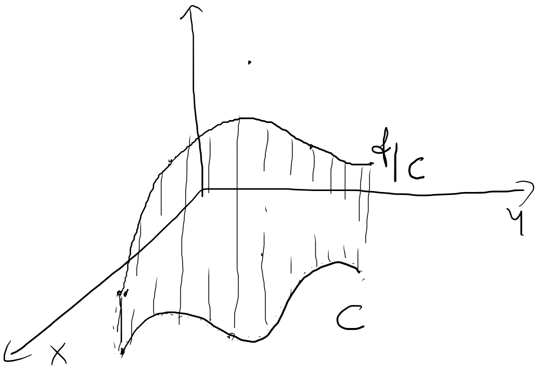
$B = \varphi(b)$



$\mathcal{C}$: $S(a_0, b_0)$ polom. R

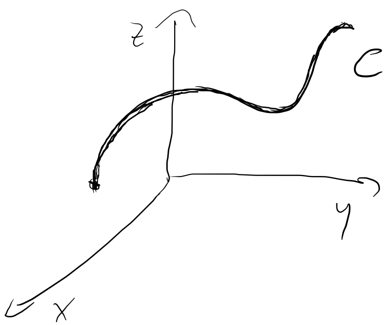
$\varphi(t) = \langle a_0 + R\cos t, \; b_0 + R\sin t \rangle \quad t \in [0, 2\pi]$





$f(x,y)$

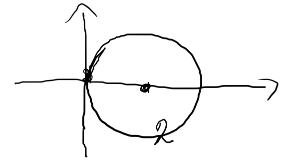
$$\int_C f(x,y)\,ds$$



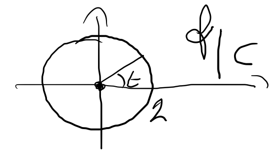
$f(x,y,z) = g(x,y,z)$

$$\int_C g(x,y,z)\,ds$$

Mass



3 Spočítejte $\int\limits_{\mathcal{C}} \frac{x+2}{\sqrt{x^2+y^2}}\, ds$, kde $\mathcal{C}$ je: a) $x^2+y^2=4x$, b) $x^2+y^2=4$.



Funkce je definovaná a spojtá v $\mathbb{R}^2 \setminus [0,0]$.

a) $[0,0] \in \mathcal{C}$, a $\lim\limits_{(x,y)\to(0,0)} \frac{x+2}{\sqrt{x^2+y^2}} = \infty$, takže integrál neexistuje.

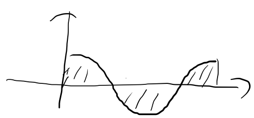
b) $[0,0] \notin \mathcal{C}$, takže integrál existuje.

$$\varphi(t) = (\overbrace{2\cos t}^{x}, \overbrace{2\sin t}^{y}) \quad \text{pro} \quad 0 \le t \le 2\pi .$$

$$\varphi'(t) = (-2\sin t, 2\cos t) \quad \text{a} \quad ||\varphi'(t)|| = \sqrt{(-2\sin t)^2 + (2\cos t)^2} = 2.$$

$$\int\limits_{\mathcal{C}} \frac{x+2}{\sqrt{x^2+y^2}}\, ds = \int\limits_0^{2\pi} \frac{2\cos t + 2}{\cancel{2}}\, \cancel{2}\, dt = 4\pi\ .$$

$$= \int_0^{2\pi} \underbrace{2\cos t\, dt}_{0} + \int_0^{2\pi} 2\, dt$$



Připomenutí: Integrál z vektorového pole $\vec{F}$ podél dané orientované křivky $\mathcal{C}$ počítáme jako práci síly podél této křivky s normovaným tečným polem $\vec{T}$ (jež určuje orientaci křivky $\mathcal{C}$).

Jestliže parametrizace $\varphi : \langle a, b\rangle \to \mathcal{C}$ odpovídá zvolené orientaci, pak máme

$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{s} = \int_{\mathcal{C}} (\vec{F}\cdot\vec{T})ds = \int_a^b \vec{F}(\varphi(t))\cdot\varphi'(t)\ dt\ .$$

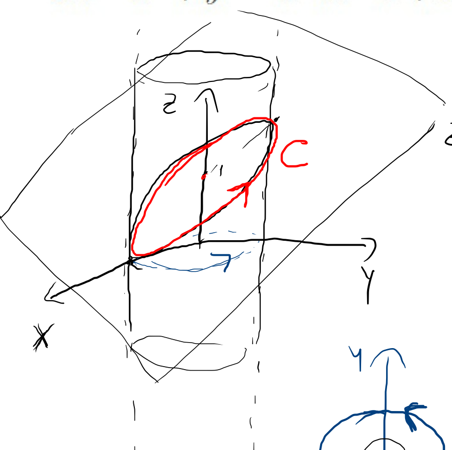
Pokud parametrizace φ je v opačném směru než námi zvolená orientace $\mathcal{C}$, $\displaystyle\int_{\mathcal{C}} \vec{F}\cdot d\vec{s} = -\int_a^b \vec{F}(\varphi(t))\cdot\varphi'(t)\ dt\ .$

Určete

$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{s},\quad \text{kde}\quad \vec{F} = (y,\ z,\ x)$$

$\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$

$\mathcal{C}:\quad x^2+y^2=1\quad \&\quad x+z=1\quad$ je křivka s kladnou orientací při pohledu shora.



$x+z=1$

$z=1-x$

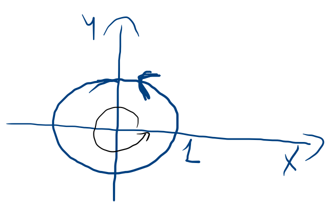
$C:\ \varphi(t) = (\cos t,\ \sin t,\ 1-\cos t)\qquad t\in\langle 0, 2\pi\rangle$

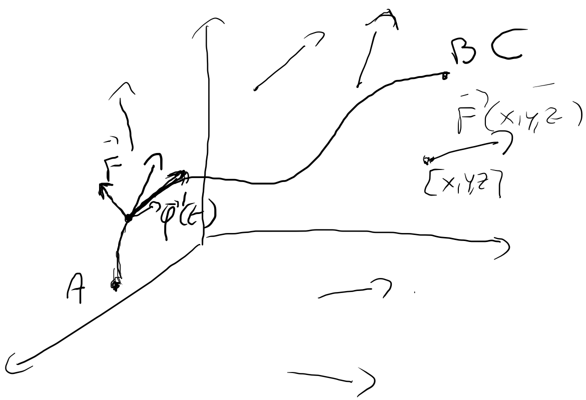
$\varphi'(t) = (-\sin t,\ \cos t,\ \sin t)$

$$\int_0^{2\pi} (\sin t,\ 1-\cos t,\ \cos t)\cdot(-\sin t,\ \cos t,\ \sin t)\ dt =$$

$$= \int_0^{2\pi} -\sin^2 t + \cos t - \cos^2 t + \sin t\cos t\ dt$$

$$= \int_0^{2\pi} -1\,dt + \underbrace{\int_0^{2\pi} \cos t\,dt}_{0} + \underbrace{\int_0^{2\pi} \sin t\cos t\,dt}_{\left[\frac{\sin^2 t}{2}\right]_0^{2\pi} = 0} = \underline{\underline{-2\pi}}$$





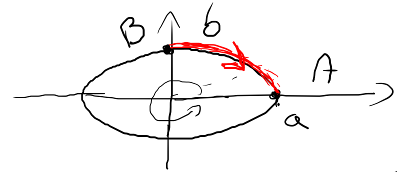
$$\vec{F} : \mathbb{R}^n \to \mathbb{R}^n \quad n = 2, 3$$

$$\int_C \vec{F} \cdot ds$$

Určete

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{s}, \quad \text{kde} \quad \vec{F} = (-y,\ x) \qquad \vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$$

$\mathcal{C}: \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ x \geq 0,\ y \geq 0,$ od $B = [0, b]$ do $A = [a, 0]$.



$$\varphi(t) = (a\cos t,\ b\sin t) \qquad t \in (0, \pi/2)$$

$$\varphi'(t) = (-a\sin t,\ b\cos t)$$

$$t \in [\pi/2, 0]$$

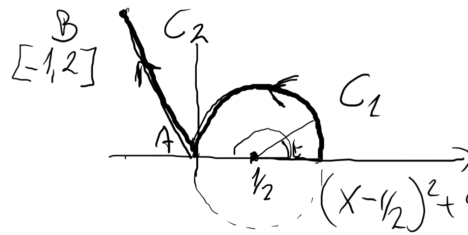
$$\varphi: \mathbb{R} \to \mathbb{R}^2$$

$$\int_C \vec{F} \cdot ds = -\int_0^{\pi/2} (-b\sin t,\ a\cos t) \cdot (-a\sin t,\ b\cos t)\, dt =$$

$$\int_{\pi/2}^{0} \dots = -\int_0^{\pi/2}$$

$$= -\int_0^{\pi/2} (ab\sin^2 t + ab\cos^2 t)\, dt = -ab \cdot \frac{\pi}{2}$$

Spočítejte $\int_{\mathcal{C}} x^2\,ds$, kde $\mathcal{C}$ je graf funkce $f(x)=\ln x$, $x\in<1,2>$. DÚ



$(x-\frac{1}{2})^2+y^2=\frac{1}{4}$

$S(\frac{1}{2},0)$ $\quad R=\frac{1}{2}$

$$\int_C \sqrt{x^2+y^2}\,ds \qquad \text{kde } C=C_1\cup C_2$$

$$\int_C f\,ds=\int_{C_1} f\,ds+\int_{C_2} f\,ds=1+\frac{5}{2}=\frac{7}{2}$$

$$C_1:\ \varphi_1(t)=(\overbrace{\tfrac{1}{2}+\tfrac{1}{2}\cos t}^{x},\ \overbrace{\tfrac{1}{2}\sin t}^{y}) \qquad \varphi_1'(t)=(-\tfrac{1}{2}\sin t,\ \tfrac{1}{2}\cos t) \qquad \|\varphi_1'(t)\|=\frac{1}{2}$$

$t\in\langle 0,\pi\rangle$

$$\int_0^\pi \sqrt{\frac{1}{4}+\frac{1}{2}\cos t+\frac{1}{4}\cos^2 t+\frac{1}{4}\sin^2 t}\cdot\frac{1}{2}\,dt=\int_0^\pi\sqrt{\frac{1}{2}+\frac{1}{2}\cos t}\,dt=1$$

$$C_2:\ \overline{AB}:\ \varphi(t)=A+t(B-A),\quad t\in\langle 0,1\rangle$$

$\varphi(t)=(-t,2t)$

$\varphi'(t)=(-1,2)$

$\|\varphi'(t)\|=\sqrt{5}$

$$\int_0^1\sqrt{t^2+4t^2}\cdot\sqrt{5}\,dt=\int_0^1 5\underbrace{\sqrt{t^2}}_{|t|}\,dt=5\left[\frac{t^2}{2}\right]_0^1=\frac{5}{2}$$